Schwarzschild metrics and quasi-universes

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Abstract

The exterior and interior Schwarzschild solutions are rewritten replacing the usual radial variable with an angular one. This allows to obtain some results otherwise less apparent or even hidden in other coordinate systems.

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Introduction: It is well known [1] that the three-dimensional space

$$d^{(3)}s^2 = \frac{dr^2}{1 - \dfrac{r^2}{R^2}} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{1}$$

of some models of closed homogeneous and isotropic universes has an especially simple geometry which can be seen best introducing a new angular coordinate $0 \leq \chi \leq \pi$ via $r = R\sin\chi$ and transforming the line element (1) into the form

$$d^{(3)}s^2 = R^2\,(d\chi^2 + \sin^2\chi\, d\Omega^2) \tag{2}$$

where

$$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2 \tag{3}$$

The metric (2) is that of a three-dimensional hypersurface of radius R which can be represented in a flat, four-dimensional Euclidean embedding space.

Our purpose is to employ a similar angular variable to describe the geometry of the exterior Schwarzschild solution and to investigate such a description of the interior solution also when $\chi > \pi/2$, a possibility which appears to have been ignored in the literature.

The exterior Schwarzschild solution: The exterior spherically symmetric vacuum solution, which by Birkhoff's theorem is also static, will be written in standard coordinates as

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2 - N^2(t)\left(1 - \frac{2m}{r}\right) dt^2 \tag{4}$$

The term $N^2(t)$ allows the matching between exterior and interior values of g_{tt} when the interior solution is not static and the observer is below the radius r_1 of the body; of course in the static cases $N^2(t)$ reduces to a constant. Such a constant shall be written as $(1 - 2m/r_0)^{-1}$ if the observer is placed at r_0 above the radius r_1; so the light will appear to him red-shifted if received from $r < r_0$ and blue-shifted if received from $r > r_0$.

Coming back to the line element (4), we want to replace the radial coordinate r with an angular coordinate ψ; because of the covariance of Einstein's equations there are infinite ways to accomplish the replacement. We choose to define an angular coordinate ψ given by

$$r = \frac{2m}{\cos^2\psi} \qquad -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2} \tag{5}$$

when $r > 2m$, and analytically continued to

$$r = \frac{2m}{\cosh^2\psi} \qquad -\infty < \psi < \infty \tag{6}$$

when $r < 2m$. The line element (4) becomes, in the region $r > 2m$

$$ds^2 = \frac{16m^2}{\cos^6\psi} d\psi^2 + \frac{4m^2}{\cos^4\psi} d\Omega^2 - \frac{\sin^2\psi}{\sin^2\psi_0} dt^2 \tag{7}$$

Here the event horizon is placed at $\psi = 0$, while infinity is reached at $\psi = \pm\pi/2$. The metrical relations in the surface $t =$ constant, $\vartheta = \pi/2$

are illustrated by means of the surface of revolution $f(r) = \sqrt{8m(r-2m)}$ (remember the representation of the Flamm's paraboloid with the Einstein-Rosen bridge). In the extended region $r < 2m$ one has instead the line element

$$ds^2 = -\frac{16m^2}{\cosh^6\psi}\,d\psi^2 + \frac{4m^2}{\cosh^4\psi}\,d\Omega^2 + \frac{\sinh^2\psi}{\sinh^2\psi_0}\,dt^2 \tag{8}$$

which describes the interior of a black hole joined to the exterior by the event horizon placed at $\psi = 0$. It is worth noticing that the introduction of the ψ coordinate provides a division of the maximally extended Schwarzschild spacetime in four regions with two singularities corresponding to an equal gravitational mass, just as described by Kruskal-Szekeres coordinates. These singularities are placed at $\psi = \pm\infty$, being now ψ a time coordinate.

The interior Schwarzschild solution: The gravitational field inside a celestial body, say a star, modelled on an ideal fluid medium with energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \tag{9}$$

is given, for static distribution of matter and pressure and moreover under the hypotheses of spherical symmetry and constant mass density, by

$$ds^2 = \frac{dr^2}{1 - \dfrac{r^2}{R^2}} + r^2 d\Omega^2 - \left[A - B\sqrt{1 - \frac{r^2}{R^2}}\right]^2 dt^2 \tag{10}$$

Here A and B are integration constants to be determined by the matching conditions. We use this simple and rather unrealistic solution as a toy model uniquely to illustrate the employ of the new angular coordinate. If we now define the angular coordinate χ as

$$\frac{r}{R} \equiv \sin\chi \qquad\qquad 0 \leq \chi \leq \pi \tag{11}$$

the line element (10) becomes

$$ds^2 = R^2\left(d\chi^2 + \sin^2\chi d\Omega^2\right) - \left[A - B\cos\chi\right]^2 dt^2 \tag{12}$$

From Einstein's equations the pressure p and the mass density ρ are

$$p = \frac{1}{8\pi R^2}\left[\frac{3B\cos\chi - A}{A - B\cos\chi}\right] \;, \qquad\qquad \rho = \frac{3}{8\pi R^2} \tag{13}$$

In formulating the matching conditions to connect the exterior and interior Schwarzschild solutions, continuity of the metric and its derivatives are to be taken into account. However in our simple example we rest on physical plausibility considerations, so we require that the metric is continuous for $\sin\chi_1 = \frac{r_1}{R}$, where r_1 is the radius of the body, that the pressure p vanishes on its surface and that the observer is in the interior at an angle χ_0.

As a result one obtains

$$\sin\chi_1 = \left(\frac{2m}{R}\right)^{1/3} \;, \quad A = \frac{3\cos\chi_1}{3\cos\chi_1 - \cos\chi_0} \;, \quad B = \frac{1}{3\cos\chi_1 - \cos\chi_0} \tag{14}$$

where m is the gravitational mass. The line element (10) can now be written

$$ds^2 = R^2 \left(d\chi^2 + \sin^2\chi d\Omega^2\right) - \left[\frac{3\cos\chi_1 - \cos\chi}{3\cos\chi_1 - \cos\chi_0}\right]^2 dt^2 \tag{15}$$

So the observer receives the frequency of light red-shifted when coming from inside and blue-shifted when coming from outside. The matching to the exterior solution requires that

$$N^2 = \left[\frac{2\cos\chi_1}{3\cos\chi_1 - \cos\chi_0}\right]^2 \left(1 - \frac{2m}{R\sin\chi_1}\right)^{-1} \tag{16}$$

If the observer is at the exterior the previous values of A and B change accordingly. The pressure becomes

$$p = \frac{3}{8\pi R^2}\left[\frac{\cos\chi - \cos\chi_1}{3\cos\chi_1 - \cos\chi}\right] \tag{17}$$

and is obviously observer independent. Because of definition (11) two cases are now to be considered, depending whether for a given value of r_1 one chooses $\chi_1 < \pi/2$ or $\chi_1 > \pi/2$. In the former case, while the mass density ρ is constant, the pressure p, which is zero at the surface, increases inwards; the solution is non singular as long as p is finite. At $r = 0$ where p takes its maximum value, this is only possible for $\chi_1^{(1)} < \arccos(1/3) \approx 0.39\pi$, that is, as known [2], for $r_1/(2m) > 9/8$. In the latter case, the pressure p takes negative values in the interior, and the solution is non singular at $r = 0$ for $\chi_1^{(2)} > \pi/2$. In both cases, the weak energy condition

$$\rho \geq 0\ , \qquad \rho + p \geq 0 \tag{18}$$

is always satisfied. We would also point out that while the surface area $S = 4\pi R^2 \sin^2\chi_1$ is the same in the two cases, independently of the choice made for χ_1, things are different in calculating volumes, given by the formula

$$V = 4\pi R^3 \int_0^{\chi_1} \sin^2\chi \, d\chi = \pi R^3 (2\chi_1 - \sin 2\chi_1) \tag{19}$$

To make an example let us consider two bodies having the same gravitational mass but different values of χ_1 given respectively by $\chi_1^{(1)}$ and $\chi_1^{(2)} = \pi - \chi_1^{(1)}$ (and so the same value of $\sin\chi_1$). The ratio $V^{(2)}/V^{(1)}$ of their volumes is

$$\frac{V^{(2)}}{V^{(1)}} = \frac{2(\pi - \chi_1^{(1)}) + \sin 2\chi_1^{(1)}}{\chi_1^{(1)} - \sin 2\chi_1^{(1)}} \tag{20}$$

Therefore while the volume $V^{(1)}$ encloses a star whose matter is endowed by the usual properties ($\rho > 0,\ p > 0$), the volume $V^{(2)}$ may be so large to be considered as a "quasi-universe", so named because it is an universe deprived of a spherical void, containing matter with unusual properties ($\rho > 0,\ p < 0$); we do not call such a matter exotic, because it satisfies the weak energy condition and so also the null energy condition [3]. The connection between a body and a quasi-universe through a suitable part of the Flamm paraboloid is schematically represented in Figure 1. A different possibility is shown in Figure 2 where now two quasi-universes are joined through an Einstein-Rosen bridge (with throat at $\psi = 0$) which can be renamed "extreme wormhole"; here the matching conditions to be fulfilled for the second junction are the same already seen for the first, analogous quantities being now renamed with the same letter primed. Because the throat is in the vacuum, the null energy condition is not violated; so, according to the Morris-Thorne analysis [5] it is not seen as traversable by an observer placed in a fixed forwarding station. The Einstein-Rosen bridge (or extreme wormhole) can also be considered as a limiting case, when the post-Newtonian parameter $\gamma \to 1^{+}$, of the corresponding Brans-Dicke solution [6]. Finally, because of the necessary equality of the gravitational masses in the three joined solutions, one obtains the following relation between the densities of the two quasi-universes ρ and $\rho_1^{'}$:

$$\frac{\rho}{\rho_1^{'}} = \left(\frac{\sin\chi_1}{\sin\chi_1^{'}}\right)^2 \tag{21}$$

Conclusions: The metrics corresponding to the exterior and interior Schwarzschild solutions have been rewritten replacing the usual radial coordinate with an angular one. With respect to the exterior solution, it covers four different regions of the space-time. With respect to the interior solution, it has been extended from the case $\chi < \pi/2$ (first-type solution) to the case $\chi > \pi/2$ of a quasi-universe (second-type solution). A second-type solution can be joined either to a first-type or to a second-type solution respectively through a suitable part of the Flamm's paraboloid or through a particular Einstein-Rosen bridge (extreme wormhole) provided the gravitational masses are equal.

Let us now consider Equations (7) and (8) in the limiting case when the exterior Schwarzschild solution goes over all the remaining space (asymptotic flatness). It is our opinion that the following unions ($\bigcup$) of two of the four regions - named hereafter I, II, III, IV according to the customary nomenclature [4] - of the Kruskal-Szekeres diagram give rise to distinct solutions:

1) $I \bigcup II$: there is a singularity corresponding to a gravitational mass m at $\psi = -\infty$ and a quasi-universe of gravitational mass m and density $\rho = 0$ at the boundary $\psi = \pi/2$. The two regions are separated at $\psi = 0$ by an event horizon.

2) $III \bigcup IV$: there is a singularity corresponding to a gravitational mass m at $\psi = +\infty$ and a quasi-universe of gravitational mass m and density $\rho = 0$ at the boundary $\psi = -\pi/2$. The two regions are separated at $\psi = 0$ by an event horizon.

3) $I \bigcup III$: there are two quasi-universes with gravitational mass m and density $\rho = 0$ at the boundaries $\psi = \pi/2$ and $\psi = -\pi/2$, connected by an extreme wormhole.

4) $II \bigcup IV$: the universe consists of two equal masses placed respectively at $\psi = -\infty$ and at $\psi = +\infty$ with a cosmological horizon at $\psi = 0$.

The Penrose diagram for the maximally extended Schwarzschild spacetime is a representation of the set of the four solutions.

More in general, one could consider expanding quasi-universes, which are universes with cavities [7],[8],[9],[10]. Inside one of such voids there is a body whose inertial mass is, by the equivalence principle, equal to its own gravitational mass and consequently, broadening the above considerations, also to the gravitational mass of the quasi-universe. Otherwise stated, the inertial mass of a body could be equal to the gravitational mass of its surrounding quasi-universe. While for a closed universe the angular momentum is "undefined and undefinable" [11], this is not true for a quasi-universe, and therefore it would be interesting to investigate whether its angular momentum is equal and opposite to that of the body in the cavity. As an example, the Kerr-dS solution does represent an universe containing two equal masses with two equal and opposite angular momenta [12]. If this turns out to be the general case and if one remembers what stated before about the equality of the inertial mass of the body to the gravitational mass of the quasi-universe, it will prove attractive to search a link between these facts and Mach's principle.

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Figure captions

Figure 1: The connection between a body and a quasi-universe.

Figure 2: The connection between two quasi-universes.

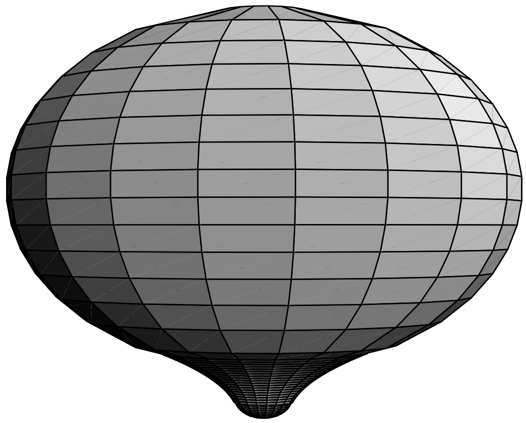
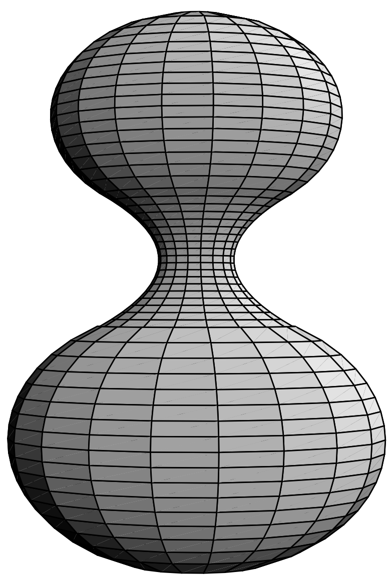


Figure 1: The connection between a body and a quasi-universe.



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